Calculus Problem Set

1. Give an ϵ, δ proof for $\lim_{x \to c} \sqrt{x} = \sqrt{c}$ for $c > 0$.

2. If $g(x) = \begin{cases} x, & x \text{ rational}, \\ 0, & x \text{ irrational}. \end{cases}$ Give an ϵ, δ proof for $\lim_{x \to 0} g(x) = 0$.

3. Find the limit.

   (a) $\lim_{x \to 1} \dfrac{x^2 - 1}{\sqrt{2x + 2} - 2}$

   (b) $\lim_{x \to \pi} (x - \pi) \cos^2 \left(\dfrac{1}{x - \pi} \right)$

   (c) $\lim_{x \to 0} x^2 \left(1 + \cot^2 3x\right)$

   (d) $\lim_{x \to 0} \left(\dfrac{1 + 2^x}{2} \right)^{1/x}$

   (e) $\lim_{x \to \pi/2} \dfrac{\ln(\sin x)}{(\pi - 2x)^2}$

   (f) $\lim_{x \to \pi/4} \dfrac{\sec^2 x - 2 \tan x}{1 + \cos 4x}$

   (g) $\lim_{x \to \infty} (x^3 + 1)^{1/\ln x}$

4. Show by example that $\lim_{x \to c} [f(x) + g(x)]$ can exist even if $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ do not exist.

5. Determine whether or not $f(x) = \begin{cases} \dfrac{|x - 1|}{x - 1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$ is continuous at $x = 1$. If not, determine whether the discontinuity is a removable discontinuity, a jump discontinuity, or an infinite discontinuity.

6. Find A and B given that the function $f(x) = \begin{cases} x^3, & x \leq 1 \\ Ax + B, & x > 1. \end{cases}$ is differentiable at $x = 1$.

7. Find the derivative by the limit process. $f(x) = \dfrac{1}{x^2}$

8. Find equations for all the lines tangent to the graph of $f(x) = x^3 - x$ that pass through the point $(-2, 2)$.

9. Find the indicated derivative. $\dfrac{d}{du} \left(\dfrac{u}{u - 1} - \dfrac{u}{u + 1} \right)$

10. Find dy/dx. $y = \tan\left(\sec x^3\right)$.

11. Find dy/dx at $x = 2$. $y = \left(\sqrt{x^2 - 3} + 3\right)^2$

12. Evaluate $\dfrac{dy}{dx}$ at the point $(1, 1)$ for $x \sin(xy - y^2) = x^2 - 1$.

13. Find the second derivative. $y = \dfrac{\cos x}{1 + \sin x}$

14. Express $\dfrac{d^2y}{dx^2}$ in terms of x and y for $4 \tan y = x^3$

15. Find the derivative.

(a) $\dfrac{d}{dx}\left(\displaystyle\int_{5^x}^{\sin^{-1} x} \sqrt{4 + t^6}\ dt \right)$

(b) $\dfrac{d}{dx}\left[x^{(e^x)} \right]$

(c) $\dfrac{d}{dx}\left[(\ln x)^{\ln x} \right]$

16. Show that $f(x) = \displaystyle\int_1^{2x} \sqrt{16 + t^4}\ dt$ has an inverse and find $(f^{-1})'(0)$.

17. Find the critical numbers, the local extreme values, and the points of inflection of $f(x) = x(x-1)^{1/5}$.

18. Find the vertical and horizontal asymptotes for $f(x) = \dfrac{2x}{\sqrt{x^2 - 1}}$

19. Show that the equation $6x^5 + 13x + 1 = 0$ has exactly one real root.

20. Given that f and g are continuous functions on $[a, b]$, and that $f(a) < g(a)$ and $g(b) < f(b)$, show that there exists at least one number $c \in (a, b)$ such that $f(c) = g(c)$.

21. Given that $|f'(x)| \leq 1$ for all real numbers x, show that $|f(x_1) - f(x_2)| \leq |x_1 - x_2|$ for all real numbers x_1 and x_2.

22. Use differentials to estimate $\sqrt[3]{1002}$.

23. The volume of a spherical balloon is increasing at a constant rate of 8 cubic feet per minute. How fast is the radius increasing when the radius is exactly 10 feet? How fast is the surface area increasing at that instant?

24. A 13-foot ladder is leaning against a vertical wall. If the bottom of the ladder is being pulled away from the wall at the rate of 2 feet per second, how fast is the area of the triangle formed by the wall, the ground, and the ladder changing at the instant the bottom of the ladder is 12 feet from the wall?

25. Evaluate $\displaystyle\int_0^3 \left[\frac{d}{dx}\left(\sqrt{4 + x^2} \right) \right] dx$

26. Find or evaluate the integral.

(a) $\displaystyle\int \frac{\sqrt{x}}{1 + x\sqrt{x}}\ dx$

(b) $\displaystyle\int \frac{\log_2(x^3)}{x}\ dx$

(c) $\displaystyle\int \frac{\sec^2 x}{9 + \tan^2 x} dx$

(d) $\displaystyle\int \sec^3 x\ dx$

(e) $\displaystyle\int \frac{x}{\sqrt{x^2 - 2x - 3}}\ dx, \quad x > 3$

(f) $\displaystyle\int \frac{x}{x^3-1}\,dx$

(g) $\displaystyle\int \frac{dx}{(x-1)(x^2+1)^2}$

(h) $\displaystyle\int \frac{dx}{x^4+4}$

(i) $\displaystyle\int \frac{e^{2x}}{2e^{2x}+3}dx$

(j) $\displaystyle\int x\sqrt{6x-x^2-8}\,dx$

(k) $\displaystyle\int \frac{1}{1+\sin x+\cos x}\,dx$

(l) $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos x}{1+\sin x}dx$

(m) $\displaystyle\int_{0}^{\pi/2} \frac{1}{1+\sin x}\,dx$

(n) $\displaystyle\int_{1}^{4} \frac{dx}{x^2-4}$

(o) $\displaystyle\int_{0}^{1} x10^{1+x^2}dx$

(p) $\displaystyle\int_{0}^{1} x\ln x\,dx$

(q) $\displaystyle\int_{1}^{\infty} \frac{dx}{x(x+1)}$

(r) $\displaystyle\int_{0}^{1} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

27. Find the area of the region bounded by $x = y^2$ and $x = 12 - 2y^2$

28. The base of a solid is the triangular region bounded by the y-axis and the lines $x+2y = 4$, $x - 2y = 4$. Find the volume of the solid given that the cross sections perpendicular to the x-axis are squares.

29. Find the volume of the solid generated by revolving the region bounded by the curves of $y = \sqrt{x}$, and $y = x^3$ about the x-axis.

30. Find the volume of the solid generated by revolving the region bounded by $y = x\cos x$, $x \in [0, \frac{\pi}{2}]$ about the y-axis.

31. Use the shell method to find the volume of the solid generated by revolving the region bounded by the graphs of $y = \sqrt{x}$, and $y = x^3$ about the y-axis.

32. Find the area of the surface generated by revolving $f(x) = \frac{1}{3}x^3$, $x \in [0, 2]$ about the x-axis.

33. Determine the boundedness and monotonicity of the sequence $\left\{\ln\left(\dfrac{2n}{n+1}\right)\right\}$.

34. State whether or not the sequence $\left(\dfrac{n+1}{n+2}\right)^n$ converges as $n \to \infty$, and, if it does, find the limit.

35. Determine the convergence or divergence of the series. Identify the test used.

(a) $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+1}$

(b) $\sum_{k=1}^{\infty} \frac{k^k}{(3^k)^2}$

(c) $\sum_{k=1}^{\infty} xe^{-k^2}$

(d) $\sum_{k=1}^{\infty} \frac{k^{k/2}}{k!}$

(e) $\sum_{k=2}^{\infty} \frac{1}{k}\left(\frac{1}{\ln k}\right)^{1/2}$

(f) $\sum_{k=1}^{\infty} \frac{2k+1}{\sqrt{k^5+1}}$

(g) $\sum_{k=1}^{\infty} (-1)^k k \sin(1/k)$

36. Test the series $\sum_{k=1}^{\infty} (-1)^k \frac{k}{k^2+1}$ for (a) absolute convergence, (b) conditional convergence.

37. Find the interval of convergence of $\sum_{k=1}^{\infty} (-1)^k \left(\frac{2}{3}\right)^k (x+1)^k$

38. Find the interval of convergence. $\sum_{k=1}^{\infty} \frac{1}{k2^k} x^k$

39. Use the definition to find the Maclaurin series for $f(x) = \sin x$.

40. Use the definition to find the Taylor series for $f(x) = \ln x$ at $c = 1$.

41. Find $f^{(9)}(0)$ for $f(x) = x\cos(x^2)$.

42. Find a power series representation for $\int_0^x \frac{\ln(1+t)}{t} dt$.

43. Expand $\frac{1}{\sqrt{1+x}}$ in power of x up to x^4.

44. Identify the curve given by $r = \frac{4}{1-\cos\theta}$ and write the equation in rectangular coordinates.

45. Find the area of the region that is outside $r = \cos 2\theta$ but inside $r = 1$, and sketch the polar curves.

46. Find the area of the region enclosed by the inner loop of $r = 1 - 2\sin\theta$ and sketch the polar curve.

47. Find the points (x, y) at which the curve $x(t) = \cos t$, $y(t) = \sin 2t$ has a vertical tangent.

48. Find the length of the polar curve $r = e^{2\theta}$ from $\theta = 0$ to $\theta = 2\pi$.