# NUK Math 徵答003 解答

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**問題003**

If $f(x) = \frac{\sin nx}{n^p}$, for $n = 1, 2, 3, \cdots$ and $p \in \mathbb{R}$. Prove that

**(i)** the sequence $\{f_n\}$ converges uniformly on $\mathbb{R}$ iff $p > 0$.

**(ii)** the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $\mathbb{R}$ iff $p > 1$.

**(i)** 吳士逸的解法

($\Rightarrow$) Suppose $p \leq 0$.

Case1: $p = 0, f_n(x) = \sin nx$. When $x = \frac{\pi}{2}$, we consider $m_n = 2n$, $k_n = 4n + 1$. Then take $n \to \infty$, we have $m_n \to \infty$ and $k_n \to \infty$, this implies $f_{m_n}(x) = 0$ and $f_{k_n}(x) = 1$. Thus $f_n$ does not converge pointwise at $x = \frac{\pi}{2}$. Hence $f_n$ is not converge uniformly on $\mathbb{R}$.

Case2: $p < 0$.

If $p < 0$, we set $k = -p$, wirte $f_n(x) = \frac{\sin nx}{n^p} = n^{-p} \sin nx = n^k \sin nx$, where $k > 0$. Take $x = \frac{\pi}{2}$ and $m = 4n + 1$, $n \in \mathbb{N}$. Then $f_m(x) = (4n + 1)^k \sin(2n\pi + \frac{\pi}{2}) = (4n + 1)^k$. Since $n \in \mathbb{N}$ and $k > 0$ is given, if $n$ large enough, $f_n$ diverge at $x = \frac{\pi}{2}$. Hence $f_n$ is not uniformly converge on $\mathbb{R}$.

($\Leftarrow$) Given $p > 0$, then $\forall \varepsilon > 0$, if $n$ large enough, we have $|\frac{\sin nx}{n^p} - 0| \leq |\frac{1}{n^p}| < \varepsilon$ for all $x \in \mathbb{R}$. Then by $M$-test, $f_n$ converge uniformly to 0.

**(ii)** ($\Leftarrow$) Suppose that $p > 1$. Since $\sum |f_n| \leq \sum \frac{1}{n^p}$ and $\sum \frac{1}{n^p}$ is $p$-series with $p > 1$, it is converge. Then by $M$-test, $\sum f_n$ converge uniformly on $\mathbb{R}$.

($\Rightarrow$) Suppose that $\sum f_n$ converge uniformly on $\mathbb{R}$. Then for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n > m > N$,

$$\left| \sum_{k=m}^{n} f_k(x) \right| = \left| \sum_{k=m}^{n} \frac{\sin kx}{k^p} \right| < \frac{4\varepsilon}{\sqrt{2}} \text{ for all } x \in \mathbb{R}.$$

Thus, when $n > 2N$, we have $\lfloor \frac{n}{2} \rfloor \geq N$, where

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd,} \end{cases}$$

this implies

$$\left| \sum_{k=\lfloor \frac{n}{2} \rfloor + 1}^{n} \frac{\sin kx}{k^p} \right| < \frac{4\varepsilon}{\sqrt{2}} \text{ for all } x \in \mathbb{R}.$$

Take $x = \frac{\pi}{2n}$. Since $\lfloor \frac{n}{2} \rfloor + 1 > \frac{n}{2}$ for all $n$,

$$\sin\frac{\pi}{4} \leq \sin kx \leq \sin\frac{\pi}{2} \text{ for all } k = \left\lfloor \frac{n}{2} \right\rfloor + 1, \ldots, n.$$

Thus,

$$\sum_{k=\lfloor \frac{n}{2} \rfloor + 1}^{n} \frac{\sin kx}{k} \geq \sin\frac{\pi}{4} \sum_{k=\lfloor \frac{n}{2} \rfloor + 1}^{n} \frac{1}{n^p} \geq \frac{\sqrt{2}\, n}{4} \frac{1}{n^p}.$$

We can conclude that if $\sum f_n$ converge uniformly on $\mathbb{R}$, then for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n > 2N$, we have

$$\frac{1}{n^{p-1}} < \varepsilon \text{ or } \lim_{n\to\infty} \frac{1}{n^{p-1}} = 0,$$

this implies $p > 1$.

**答題優良名單**:
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