NUK Math 徵答 006 解答

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問題 006

$x, y \in \mathbb{R}^n$ 為單位向量，令 $H(\alpha) = I - \alpha x y^T$

(1) Show that if $\alpha y^T x \neq 1$ then $H(\alpha)$ is invertible.

(2) Find $H(\alpha)^{-1}$ when $\alpha y^T x \neq 1$

解 1 :翁嘉駿及謝偉勃的解法

當 $\alpha y^T x \neq 1$ 時，令 $\beta = \dfrac{\alpha}{\alpha y^T x - 1}$

$$
\begin{aligned}
H(\alpha)H(\beta) &= (I - \alpha x y^T)(I - \beta x y^T) \\
&= I - (\alpha + \beta) x y^T + \alpha\beta x y^T x y^T \\
&= I - (\alpha + \beta) x y^T + \alpha\beta (y^T x) x y^T \\
&= I - (\alpha + \beta - \alpha\beta (y^T x)) x y^T \\
&= I
\end{aligned}
$$

因此，當 $\alpha y^T x \neq 1$ 時 $H(\alpha)$ 可逆，且 $H(\alpha)^{-1} = I - \dfrac{\alpha}{\alpha y^T x - 1} x y^T$ 。

解２ :連威翔的解法

ⓐ Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$, then

$$H(\alpha) = I - \alpha x y^T = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} - \alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} [y_1 \ y_2 \cdots y_N]$$

$$= \begin{bmatrix} 1-\alpha x_1 y_1 & -\alpha x_1 y_2 & \cdots & -\alpha x_1 y_N \\ -\alpha x_2 y_1 & 1-\alpha x_2 y_2 & & \vdots \\ \vdots & & \ddots & \\ -\alpha x_N y_1 & -\alpha x_N y_2 & \cdots & 1-\alpha x_N y_N \end{bmatrix}$$

Take determinant of $H(\alpha)$, we have

$$|H(\alpha)| = \begin{vmatrix} 1-\alpha x_1 y_1 & -\alpha x_1 y_2 & \cdots & -\alpha x_1 y_N \\ -\alpha x_2 y_1 & 1-\alpha x_2 y_2 & & -\alpha x_2 y_N \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha x_N y_1 & -\alpha x_N y_2 & \cdots & 1-\alpha x_N y_N \end{vmatrix}$$

Since x, y are unit vector, we have $\sum_{i=1}^{N} x_i^2 = 1$, which means at least one x_i is not zero.

Case 1 : If $x_1 = 1$ and $x_i = 0$, $2 \le i \le N$, then we have

$$|H(\alpha)| = \begin{vmatrix} 1-\alpha y_1 & -\alpha y_2 & -\alpha y_3 & \cdots & -\alpha y_N \\ 0 & 1 & 0 & & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

(Do row operations : $R_i \times (\alpha y_i) + R_1$, where $2 \le i \le N$.)

$$= \begin{vmatrix} 1-\alpha y_1 & & & \\ & 1 & & \\ & & 1 \ddots & \\ & & & 1 \end{vmatrix} = 1 - \alpha y_1$$

$$= 1 - \sum_{i=1}^{N} \alpha x_i y_i = 1 - \alpha y^T x \neq 0.$$ So $H(\alpha)$ is invertible.

Case 2: If $x_k \neq 0$ for some $2 \le k \le N$, then we have

$$|H(\alpha)| = \begin{vmatrix} 1-\alpha x_1 y_1 & -\alpha x_1 y_2 & \cdots & -\alpha x_1 y_N \\ -\alpha x_2 y_1 & 1-\alpha x_2 y_2 & & \vdots \\ \vdots & & \ddots & \\ -\alpha x_k y_1 & -\alpha x_k y_2 & \ddots & -\alpha x_k y_N \\ \vdots & & \ddots & \vdots \\ -\alpha x_N y_1 & -\alpha x_N y_2 & \cdots & 1-\alpha x_N y_N \end{vmatrix}$$

We do $\underset{(\text{row})}{\wedge}$ operations as : $R_k \times \left(\frac{-x_i}{x_k}\right) + R_i$, $1 \le i \le N$, $i \neq k$.

Then we have

$$
|H(\alpha)| = \begin{vmatrix}
1 & 0 & \cdots & -\frac{x_1}{x_k} & \cdots & 0 \\
0 & 1 & & -\frac{x_2}{x_k} & & \vdots \\
\vdots & \vdots & \ddots & \vdots & & \vdots \\
0 & 0 & & -\frac{x_{k-1}}{x_k} & & 0 \\
-\alpha x_k y_1 & -\alpha x_k y_2 & \cdots & 1-\alpha x_k y_k & \cdots & -\alpha x_k y_N \\
0 & 0 & & -\frac{x_{k+1}}{x_k} & 1 & 0 \\
\vdots & \vdots & & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -\frac{x_N}{x_k} & 0 \cdots & 1
\end{vmatrix}
$$

Then we do row operations : $R_i \times (\alpha x_k y_i) + R_k$, $1 \leq i \leq N$, $i \neq k$.

And we have

$$
|H(\alpha)| = \begin{array}{c|cccccc|l}
 & 1 & 0 & \cdots & -\frac{x_1}{x_k} & \cdots & 0 & \\
 & 0 & 1 & & \vdots & & 0 & \\
 & \vdots & \vdots & \ddots & -\frac{x_{k-1}}{x_k} & & \vdots & \\
 & 0 & 0 & \cdots 0 & 1-\sum_{i=1}^{N}\alpha x_i y_i & \cdots & 0 & \leftarrow \text{row } k \\
 & \vdots & \vdots & & \vdots & \ddots & \vdots & \\
 & 0 & 0 & & \vdots & & \vdots & \\
 & 0 & 0 & \cdots & -\frac{x_N}{x_k} & \cdots & 1 &
\end{array}
$$

Since $1-\alpha y^T x = 1-\sum_{i=1}^{N}\alpha x_i y_i \neq 0$, we have

$$
|H(\alpha)| = (1-\alpha y^T x) \cdot \begin{array}{|cccc|l}
1 & 0 & -\frac{x_1}{x_k} & \cdots 0 & \\
0 & 1 & \vdots & \vdots & \\
\vdots & \ddots & 1 & \vdots & \leftarrow \text{row } k \\
\vdots & & \vdots & \ddots & \\
0 & \cdots & -\frac{x_N}{x_k} & 1 &
\end{array}
$$

Then we do row operations : $R_k \times \left(-\dfrac{x_i}{x_k}\right) + R_i$, $1 \le i \le N$, $i \ne k$.

And we have

$$|H(\alpha)| = (1 - \alpha y^T x) \cdot \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix} = 1 - \alpha y^T x \ne 0.$$

Hence $H(\alpha)$ is invertible.

By Case 1, 2, we know $H(\alpha)$ is invertible if $\alpha y^T x \ne 1$.

(b) When $\alpha y^T x \ne 1$, consider the matrix $[H(\alpha) | I]$ as

$$\left[\begin{array}{ccccc|ccc} 1-\alpha x_1 y_1 & -\alpha x_1 y_2 & \cdots & \cdots & -\alpha x_1 y_N & 1 & & 0 \\ -\alpha x_2 y_1 & 1-\alpha x_2 y_2 & & & \vdots & 0 & \ddots & \\ \vdots & & \ddots & & \vdots & 0 & & \\ -\alpha x_k y_1 & & 1-\alpha x_k y_k & & \vdots & \vdots & & \\ \vdots & & & \ddots & -\alpha x_{N-1} y_N & 0 & & \\ -\alpha x_N y_1 & \cdots & \cdots & \cdots & 1-\alpha x_N y_N & 0 & & 1 \end{array}\right]$$

Case 1 : If $x_1 = 1$ and $x_i = 0$ for $2 \le i \le N$, then $[H(\alpha) | I]$ is

$$\left[\begin{array}{cccc|ccc} 1-\alpha y_1 & -\alpha y_2 & \cdots & -\alpha y_N & 1 & & 0 \\ 0 & 1 & & \vdots & & \ddots & \\ \vdots & \vdots & \ddots & \vdots & & & \\ 0 & 0 & & 1 & 0 & & 1 \end{array}\right]$$

Like ⓐ, we do row operations including (1), (2) as:

(1) $R_i \times (\alpha y_i) + R_1, \quad 2 \le i \le N$

(2) $R_1 \times (1-\alpha y^T x)^{-1}$. $\quad (1-\alpha y^T x = 1-\alpha y_1.)$

Let $\Delta = 1-\alpha y^T x$, then we get

$$\left[\begin{array}{cccc|ccccc} 1 & & & & \frac{1}{\Delta} & \frac{\alpha y_2}{\Delta} & \frac{\alpha y_3}{\Delta} & \cdots & \frac{\alpha y_N}{\Delta} \\ & 1 & & & & 1 & & & \\ & & \ddots & & & & 1 & & \\ & & & 1 & & & & \ddots & 1 \end{array}\right].$$

Hence $H(\alpha)^{-1}$

$$= \begin{bmatrix} \frac{1}{\Delta} & \frac{\alpha y_2}{\Delta} & \cdots & \frac{\alpha y_N}{\Delta} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Case 2: If $x_k \ne 0$ for some $2 \le k \le N$, then we do row operations including (1), (2), (3), (4) as:

(1) $R_k \times \left(\frac{-x_i}{x_k}\right) + R_i, \quad 1 \le i \le N, \quad i \ne k;$

(2) $R_i \times (\alpha x_k y_i) + R_k, \quad 1 \le i \le N, \quad i \ne k;$

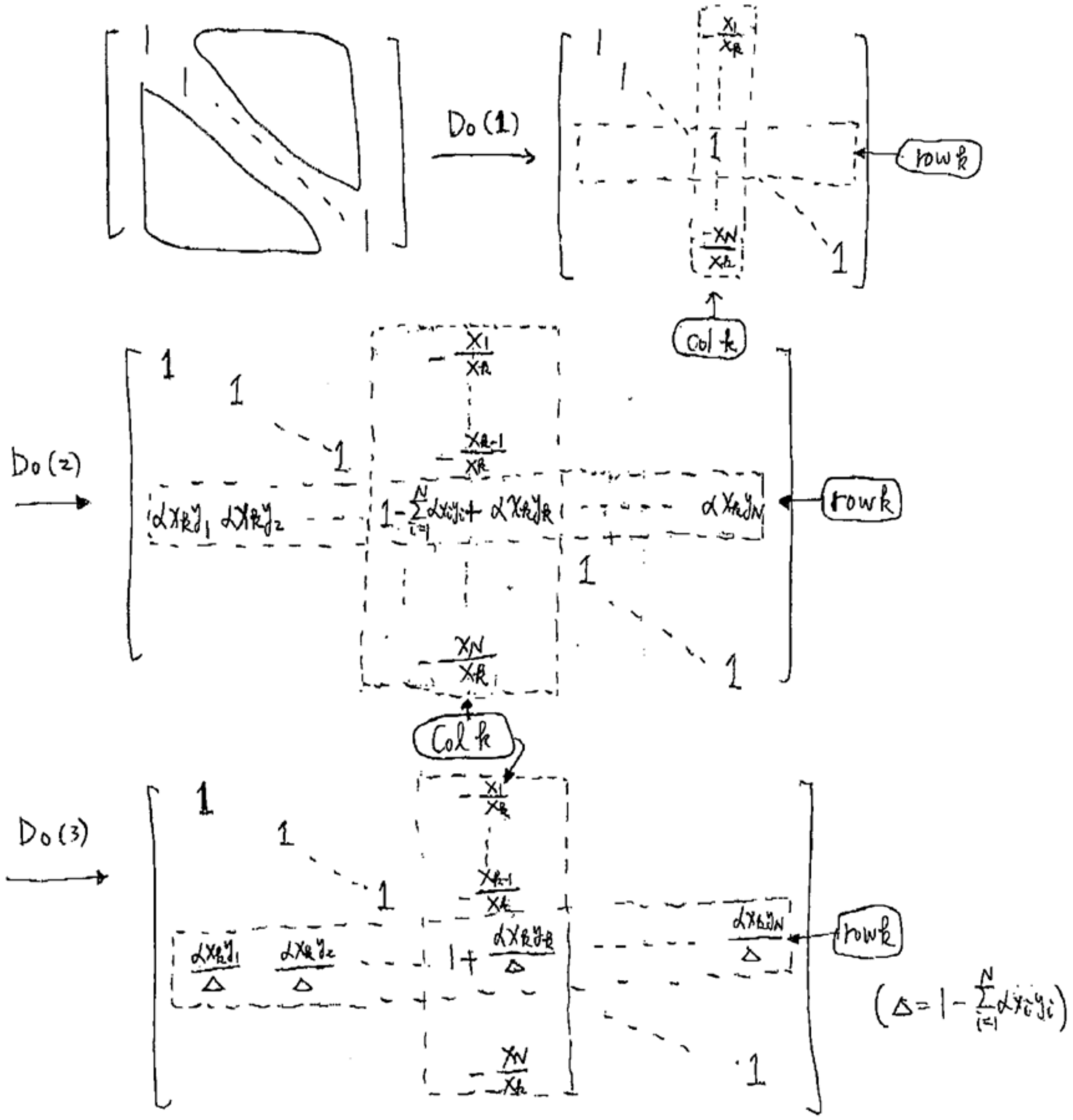
(3) $R_k \times (1-\alpha y^T x)^{-1}; \quad \left(1-\alpha y^T x = 1-\sum_{i=1}^{N} \alpha x_i y_i\right)$

(4) $R_k \times \left(\frac{x_i}{x_k}\right) + R_i, \quad 1 \le i \le N, \quad i \ne k.$

After (1), (2), (3), (4), $H(\alpha)$ becomes I, and I becomes $H(\alpha)^{-1}$.

$$\left[H(\alpha) \,\middle|\, I \right] \xrightarrow[\text{(3), (4)}]{\text{Do (1), (2)}} \left[I \,\middle|\, H(\alpha)^{-1} \right]$$

The followings show how I becomes $H(\alpha)^{-1}$:



$$\xrightarrow{D_0(4)} \begin{bmatrix} 1+\frac{\alpha x_1 y_1}{\Delta} & \frac{\alpha x_1 y_2}{\Delta} & \cdots & \frac{\alpha x_1 y_k}{\Delta} & \cdots & \frac{\alpha x_1 y_N}{\Delta} \\ \vdots & 1+\frac{\alpha x_2 y_2}{\Delta} & \ddots & \vdots & & \vdots \\ \frac{\alpha x_k y_1}{\Delta} & \frac{\alpha x_k y_2}{\Delta} & \cdots & 1+\frac{\alpha x_k y_k}{\Delta} & \cdots & \frac{\alpha x_k y_N}{\Delta} \\ \vdots & & & \vdots & \ddots & \vdots \\ \frac{\alpha x_N y_1}{\Delta} & \cdots & \cdots & \frac{\alpha x_N y_k}{\Delta} & \cdots & 1+\frac{\alpha x_N y_N}{\Delta} \end{bmatrix} = I + \frac{\alpha}{\Delta} x y^T$$

$\vdots \ldots (*)$

In case 1, since $\frac{1}{\Delta} = 1 + \frac{1}{\Delta} - 1 = 1 + \frac{1-\Delta}{\Delta} = 1 + \frac{1-(1-\alpha y_1)}{\Delta} = 1 + \frac{\alpha y_1}{\Delta}$,

$$H(\alpha)^{-1} = \begin{bmatrix} 1+\frac{\alpha y_1}{\Delta} & \frac{\alpha y_2}{\Delta} & \cdots & \frac{\alpha y_N}{\Delta} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = I + \frac{\alpha}{\Delta} x y^T \ldots (\#)$$

By $(*)$, $(\#)$, we know $H(\alpha)^{-1} = I + \frac{\alpha}{\Delta} x y^T$.

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