NUK Math 徵答 008 解答：
郭岳承

問題 008
Let $M_{n\times m}(\mathbb{R})$ be the set of $n\times m$ real matrices. Suppose that $A\in M_{n\times k}(\mathbb{R})$ and $B\in M_{m\times \ell}(\mathbb{R})$. Let $T: M_{k\times m}(\mathbb{R})\to M_{n\times \ell}(\mathbb{R})$ be defined by

$$T(X)=AXB.$$

a. Show that T is a linear transformation.
b. Let $A=\begin{bmatrix}3&2\\2&0\end{bmatrix}$ and $B=\begin{bmatrix}1&0\\0&2\end{bmatrix}$. Find the eigenvalues and eigenvectors of T.

解 1： 王瓊誼及謝偉勃的解法：

a. $\forall\ X,Y\in M_{k\times m}(\mathbb{R})$, $c\in\mathbb{R}$, we have

$$\begin{aligned}T(X+cY)&=A(X+cY)B=(AX+cAY)B=AXB+cAYB\\&=T(X)+cT(Y).\end{aligned}$$

Thus, T is a linear transformation.

b. Let $\beta=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ be ordered basis for $M_{2\times 2}(\mathbb{R})$. Since

$$\begin{aligned}T\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)&=\begin{bmatrix}3&0\\2&0\end{bmatrix},\quad T\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)=\begin{bmatrix}0&6\\0&4\end{bmatrix},\\T\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)&=\begin{bmatrix}2&0\\0&0\end{bmatrix},\quad T\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}0&4\\0&0\end{bmatrix},\end{aligned}$$

we obtain that

$$[T]_\beta=\begin{bmatrix}3&0&2&0\\0&6&0&4\\2&0&0&0\\0&4&0&0\end{bmatrix}.$$

Now, we find the eigenvalues and eigenvectors of $[T]_\beta$. The characteristic equation of $[T]_\beta$ is

$$F(\lambda)=\det([T]_\beta)=0\Rightarrow(\lambda+1)(\lambda-4)(\lambda+2)(\lambda-8)=0.$$

Hence, $\lambda=-2,-1,4,8$ are eigenvalues of $[T]_\beta$. Let $E_\lambda=\operatorname{Ker}([T]_\beta-(\lambda)I)$. Since

$$\begin{aligned}E_{-2}&=\operatorname{span}\left\{\begin{bmatrix}0\\1\\0\\-2\end{bmatrix}\right\},\quad E_{-1}=\operatorname{span}\left\{\begin{bmatrix}1\\0\\-2\\0\end{bmatrix}\right\},\\E_{4}&=\operatorname{span}\left\{\begin{bmatrix}2\\0\\1\\0\end{bmatrix}\right\},\quad E_{8}=\operatorname{span}\left\{\begin{bmatrix}0\\2\\0\\1\end{bmatrix}\right\},\end{aligned}$$

we obtain that

- $\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$ is eigenvector corresponding to eigenvalue $\lambda = -2$.
- $\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$ is eigenvector corresponding to eigenvalue $\lambda = -1$.
- $\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$ is eigenvector corresponding to eigenvalue $\lambda = 4$.
- $\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$ is eigenvector corresponding to eigenvalue $\lambda = 8$.

解 2： 求解 T 的 eigenvalues and eigenvectors 有一個比較快速的算法：
令 $\lambda_1^A = -1,\ \lambda_2^A = 4,\ \lambda_1^B = 1,\ \lambda_2^B = 2,\ v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$$Av_1 = \lambda_1^A v_1, \quad Av_2 = \lambda_2^A v_2, \quad u_1^\top B = \lambda_1^B u_1^\top, \quad u_2^\top B = \lambda_2^B u_2^\top.$$

It is easy to check that

$$T(v_i u_j^\top) = A(v_i u_j^\top)B = \lambda_i^A \lambda_j^B (v_i u_j^\top), \text{ for } 1 \leqslant i, j \leqslant 2.$$

Hence, $v_i u_j^\top$ is eigenvector corresponding to eigenvalue $\lambda_i^A \lambda_j^B$ for $1 \leqslant i, j \leqslant 2$.

解 3： 連威翔的解法

ⓐ. $T(X+Y) = A(X+Y)B = [A(X+Y)]B = [AX+AY]B$

$= AXB + AYB = T(X)+T(Y)$

$T(cX) = A(cX)B = [A(cX)]B = [c(AX)]B$

$= c(AXB) = cT(X)$, where $c \in \mathbb{R}$.

Hence T is a linear transformation.

ⓑ Since $A \in M_{2\times2}(\mathbb{R})$, $B \in M_{2\times2}(\mathbb{R})$, we know $X \in M_{2\times2}(\mathbb{R})$.

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$T(X) = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3a+2c & 3b+2d \\ 2a & 2b \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$= \begin{bmatrix} 3a+2c & 6b+4d \\ 2a & 4b \end{bmatrix}$.

If $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda \in \mathbb{R}$, then

$T(X) = \lambda X \Rightarrow \begin{bmatrix} 3a+2c & 6b+4d \\ 2a & 4b \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$

$\Rightarrow \begin{bmatrix} (3-\lambda)a+2c & (6-\lambda)b+4d \\ 2a-\lambda c & 4b-\lambda d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

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$$\begin{cases} (3-\lambda)a+2c=0 \quad \cdots\cdots ① \\ (6-\lambda)b+4d=0 \quad \cdots\cdots ② \\ 2a-\lambda c=0 \quad \cdots\cdots ③ \\ 4b-\lambda d=0 \quad \cdots\cdots ④ \end{cases}$$

Since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, at least one of (1) $a \neq 0$, (2) $b \neq 0$, (3) $c \neq 0$, (4) $d \neq 0$ must holds.

We discuss case by case as (1) to (4):

(1) If $a \neq 0$, consider $\lambda \times ① + 2 \times ③$, we have

$$[\lambda(3-\lambda)+2\times 2]a = 0 \Rightarrow \lambda^2 - 3\lambda - 4 = 0 \Rightarrow \lambda = 4 \text{ or } -1$$

If $\lambda = 4$, then $\begin{cases} -a+2c=0 \\ 2b+4d=0 \\ 2a-4c=0 \\ 4b-4d=0 \end{cases} \Rightarrow \begin{cases} a=2t \\ b=0 \\ c=t \\ d=0 \end{cases}$, where $t \in \mathbb{R}/\{0\}$.

The eigenvector is $\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$.

If $\lambda = -1$, then $\begin{cases} 4a+2c=0 \\ 7b+4d=0 \\ 2a+c=0 \\ 4b+d=0 \end{cases} \Rightarrow \begin{cases} a=t \\ b=0 \\ c=-2t \\ d=0 \end{cases}$, where $t \in \mathbb{R}/\{0\}$.

The eigenvector is $\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$.

(2) If $b \neq 0$, consider $\lambda \times ② + 4 \times ④$, we have

$$[\lambda(6-\lambda)+4\times 4]b=0 \Rightarrow \lambda^2-6\lambda-16=0 \Rightarrow \lambda=8 \text{ or } -2.$$

If $\lambda=8$, then $\begin{cases} -5a+2c=0 \\ -2b+4d=0 \\ 2a-8c=0 \\ 4b-8d=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=2t \\ c=0 \\ d=t \end{cases}$, where $t \in \mathbb{R}/\{0\}$.

The eigenvector is $\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$.

If $\lambda=-2$, then $\begin{cases} 5a+2c=0 \\ 8b+4d=0 \\ 2a+2c=0 \\ 4b+2d=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=t \\ c=0 \\ d=-2t \end{cases}$, where $t \in \mathbb{R}/\{0\}$.

The eigenvector is $\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$.

(3) If $c \neq 0$, consider $2 \times ① - (3-\lambda) \times ③$, we have

$$[2\times 2+\lambda(3-\lambda)]c=0 \Rightarrow \lambda^2-3\lambda-4=0 \Rightarrow \lambda=4 \text{ or } -1, \text{ same as case(1).}$$

By Case (1), we have the same eigenvectors: $\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$.

(4) If $d \neq 0$, consider $4 \times ② - (6-\lambda) \times ④$, we have

$$[4\times 4+\lambda(6-\lambda)]d=0 \Rightarrow \lambda^2-6\lambda-16=0 \Rightarrow \lambda=8 \text{ or } -2, \text{ same as case (2).}$$

By case (2), we have the same eigenvectors: $\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$.

By case (1), (2), (3), (4), we have eigenvalues and eigenvectors as:

(i). $\lambda_1 = 4$, $v_1 = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$;

(ii) $\lambda_2 = -1$, $v_2 = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$;

(iii) $\lambda_3 = 8$, $v_3 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$;

(iv) $\lambda_4 = -2$, $v_4 = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$,

where $T(v_i) = \lambda_i v_i$ for $1 \le i \le 4$.