Problem: *The eigenvalues of $A \in \mathbb{C}^{n\times n}$ lie in the union of the n disks in the complex plane*

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ij}| \right\}, \quad i = 1, \ldots, n.$$

以下是連威翔同學的解答(是正確的解答)

Let U be the union of n discs $D_i, 1 \leq i \leq n$.

If $n = 1$, then $A = (a_{11})$ has eigenvalue a_{11}, which lies in

$U = D_1 = \{z \in C \mid |z - a_{11}| \leq 0\} = \{a_{11}\}$.

For $n \geq 2$, let $A = (a_{ij}) \in C^{n\times n}$. By fundamental theorem of algebra, the characteristic polynomial $f(x) = |A - xI|$ has n roots in C.

If λ is one root of $f(x) = |A - xI|$, then λ is an eigenvalue of A. We have some eigenvector $x \in C^n, x \neq \theta$ such that $Ax = \lambda x$.

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, by $Ax = \lambda x$, we know $\sum_{j=1}^{n} a_{ij} x_j = \lambda x_i, i = 1,2,\ldots,n$ ... (1).

We can write (1) as $\sum_{j=1, j\neq i}^{n} a_{ij} x_j = (\lambda - a_{ii}) x_i, i = 1,2,\ldots,n$ ... (2)

Let $|x_t|$, as a function of t, has maximum at $t = k$, which means that $|x_i| \leq |x_k|$, for $1 \leq i \leq n$ and $i \neq k$.

If $|x_k| = 0$ then $|x_i| \leq |x_k| = 0$ for any $1 \leq i \leq n$ and $i \neq k$. That means $x_i = 0$ for any $1 \leq i \leq n$ and then $x = \theta$, which is a contradiction to $x \neq \theta$.

So $|x_k| \neq 0$. By (2), take $i = k$, we have $\sum_{j=1, j\neq k}^{n} a_{kj} x_j = (\lambda - a_{kk}) x_k$. So

$$|\lambda - a_{kk}||x_k| = \left| \sum_{j=1, j\neq k}^{n} a_{kj} x_j \right| \leq \sum_{j=1, j\neq k}^{n} |a_{kj}||x_j|,$$

$$|\lambda - a_{kk}| \leq \sum_{j=1, j\neq k}^{n} |a_{kj}| \frac{|x_j|}{|x_k|} \leq \sum_{j=1, j\neq k}^{n} |a_{kj}|, \text{ because } \frac{|x_j|}{|x_k|} \leq 1 \text{ for } j \neq k.$$

Now λ lies in the disc D_k, so it lies in U.

Since λ can be any one root of $f(x) = |A - xI|$, we know that all the eigenvalues of A lie in U

以下是我的解答，想法與前一頁的證明一樣只是寫得比較精簡些。

Proof. Let λ be an eigenvector of A and x a corresponding eigenvector, and let $|x_k| = \max |x_i|$. From $Ax = \lambda x$, the kth equation gives

$$\sum_{j=1, j\neq k}^{n} a_{kj}x_j = (\lambda - a_{kk})x_k.$$

Applying triangle inequality, we thus have

$$|\lambda - a_{kk}| \leq \sum_{j=1, j\neq k}^{n} |a_{kj}||x_j|/|x_k| \leq \sum_{j=1, j\neq k}^{n} |a_{kj}|,$$

since $|x_j|/|x_k| < 1$. It follows that λ belongs to the kth disk, D_k and hence is in the union of all disks.