NUK Math 徵答 014 解答：

問題 014

1 a. Let $A \in M_{n\times m}(\mathbb{R})$, $B \in M_{m\times n}(\mathbb{R})$. Show that if $\lambda \neq 0$ is an eigenvalue of AB then λ is an eigenvalue of BA.

b. Let $A = 3I_n - \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} [1, \cdots, 1]$. Find the eigenvalues and eigenvectors of A.

解答：連威翔及簡正軒的解法

(a)If $\lambda \neq 0$ is an eigenvalue of AB with respect to eigenvector v, then we have

$$ABv = \lambda v \qquad (1)$$

Since $\lambda \neq 0$ and $v \neq 0$, we know

$$\lambda v \neq 0 \qquad (2)$$

Multiplying B to the left of both sides of (1), we have

$$BA(Bv) = \lambda(Bv) \qquad (3)$$

If $Bv = 0$, then by (1) we have $\lambda v = ABv = A(Bv) = A0 = 0$, contradicting to (2).

So $Bv \neq 0$, and (3) tells that λ is an eigenvalue of BA, with respect to eigenvector Bv.

(b) Let $B = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} [1, \cdots, 1]$, then $A = 3I_n - B$. Assume that λ_B is an eigenvalue of B, with respect to eigenvector $v \neq 0$, then we have

$$Av = (3I_n - B)v = 3v - \lambda_B v = (3 - \lambda_B)v \qquad (4)$$

So $3 - \lambda_B$ is an eigenvalue of A, with respect to the same eigenvector v.

Note that $B = [1, \cdots, 1] \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \cdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$, then we have two cases as:

Case 1: $\lambda_B \neq 0$

If $\lambda_B \neq 0$, then by (a) we know that λ_B is also an eigenvalue of

$$C = [1, \ \cdots, \ 1]\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = [n].$$

Since $[n]w = nw$ for any vector $w \in R \setminus \{0\}$, n is the only eigenvalue of C. So we have $\lambda_B = n$.

Let $v = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $a_i \in R, \ 1 \le i \le n$ be the eigenvector of B with respect to $\lambda_B = n$,

and let $S = \sum_{i=1}^{n} a_i$, then

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \cdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [n]\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \Rightarrow \begin{bmatrix} S \\ S \\ \vdots \\ S \end{bmatrix} = \begin{bmatrix} na_1 \\ na_2 \\ \vdots \\ na_n \end{bmatrix} \Rightarrow S = na_1 = na_2 = \cdots = na_n \Rightarrow a_1 = a_2 = \cdots = a_n \ne 0$$

So the eigenvector is $v = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$.

Case 2: $\lambda_B = 0$

Let $v = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $a_i \in R, \ 1 \le i \le n$ be the eigenvector of B with respect to $\lambda_B = 0$,

and let $S = \sum_{i=1}^{n} a_i$. Then we have

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \cdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = [0]\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \Rightarrow \begin{bmatrix} S \\ S \\ \vdots \\ S \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow S = \sum_{i=1}^{n} a_i = 0 \Rightarrow a_n = -(a_1 + a_2 + \cdots + a_{n-1}) \cdot$$

So we have eigenvector $v = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ -(a_1 + a_2 + \cdots + a_{n-1}) \end{bmatrix}$ and all such kind of v

form a linear space $V = \left\{ a_1 v_1 + a_2 v_2 + \cdots + a_{n-1} v_{n-1} \middle| \; a_i \in R, \;\; 1 \le i \le n-1 \right\}$, where

$v_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -1 \end{bmatrix}, \cdots, v_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{bmatrix}$. Note that $v_1, v_2, \cdots, v_{n-1}$ are independent and generating for V, so we get $n-1$ eigenvectors $v_1, v_2, \cdots, v_{n-1}$ with respect to $\lambda_B = 0$.

Let λ_A be an eigenvalue of A. Going back to (4), we know that
$Av = \lambda_A v = (3 - \lambda_B) v$.
All possible values of λ_A are decided by all the values of λ_B.

Consider Case 1, Case 2, the eigenvalues λ_A and all the n eigenvectors v of $A = 3I_n - B$ are:

(i) $\lambda_A = 3 - n$, $v = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$; (ii) $\lambda_A = 3$, $v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -1 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{bmatrix}$.