國立高雄大學應用數學系

每月挑戰 019 解答

陳晴玉教授

2013年4月

答題優良名單： 以下是連威翔同學的解答。

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(1)$\dfrac{e^{ixt}}{1+t^2}=\dfrac{e^{ixt}}{(t+i)(t-i)}$. By residue theorem,

$\operatorname{Res}_{t=i}\dfrac{e^{ixt}}{1+t^2}=\lim\limits_{t\to i}\dfrac{e^{ixt}(t-i)}{(t+i)(t-i)}=\lim\limits_{t\to i}\dfrac{e^{ixt}}{t+i}=\dfrac{e^{-x}}{2i}$, where 'Res' means residue, and

$J(x)=\oint_C\dfrac{e^{ixt}}{1+t^2}dt=2\pi i\times\dfrac{e^{-x}}{2i}=\pi e^{-x}$.

(2)Choose a real number $R>1$ and let $K_R=\{R\cdot e^{i\theta}\mid 0\le\theta\le\pi\}$ .Choose the contour $C=C_R=\{-R<t<R\}\cup K_R$, then C enclose i.

Integrate $\dfrac{e^{ixt}}{1+t^2}$ along C, then $J(x)=\oint_C\dfrac{e^{ixt}}{1+t^2}dt=\int_{-R}^{R}\dfrac{e^{ixt}}{1+t^2}dt+\int_{K_R}\dfrac{e^{ixt}}{1+t^2}dt$.

The first integral:

$$\int_{-R}^{R}\frac{e^{ixt}}{1+t^2}dt=\int_{-R}^{R}\frac{\cos xt+i\sin xt}{1+t^2}dt=\int_{-R}^{R}\frac{\cos xt}{1+t^2}dt+i\int_{-R}^{R}\frac{\sin xt}{1+t^2}dt=2\int_0^R\frac{\cos xt}{1+t^2}dt$$

because $\dfrac{\cos xt}{1+t^2}$ is an even function and $\dfrac{\sin xt}{1+t^2}$ is odd.

Let R tends to infinity, then $\int_{-R}^{R}\dfrac{e^{ixt}}{1+t^2}dt=2\int_0^\infty\dfrac{\cos xt}{1+t^2}dt$ and

$$\left|\int_{K_R}\frac{e^{ixt}}{1+t^2}dt\right|\le\int_{K_R}\left|\frac{e^{ixt}}{1+t^2}\right||dt|=\int_{K_R}\frac{1}{|1+t^2|}|dt|\le\frac{\pi R}{R^2-1}\to 0$$

→ $t=Re^{i\theta}$ has a nonzero imaginary part, so $|e^{ixt}|\ne 1$. See solution later.

because K_R has arc-length πR, $|e^{ixt}|=1$,

and $|1+t^2|\ge|t^2|-1=|t|^2-1=R^2-1>0\Rightarrow\dfrac{1}{|1+t^2|}\le\dfrac{1}{R^2-1}$, for any $t\in K_R$.

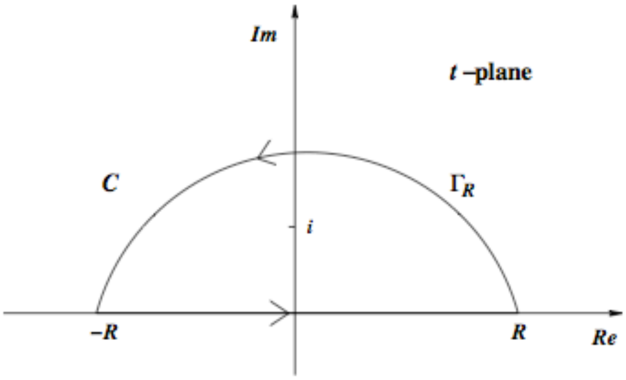
So $J(x)=2\int_0^\infty\dfrac{\cos xt}{1+t^2}dt\Rightarrow\int_0^\infty\dfrac{\cos xt}{1+t^2}dt=\dfrac{J(x)}{2}$.

以下是我的解答。

Consider contour integral

$$J(x) = \oint_C \frac{e^{\imath xt}}{1+t^2}\, dt,$$

with the contour C below



We will evaluate the integral (i) by Residue theorem and (ii) in sections.

(i) Residue theorem gives, with the contour C enclosing the singularity $t = i$ only,

$$J(x) = 2\pi\imath \times \text{Residue}_{t=\imath}\left(\frac{e^{\imath xt}}{1+t^2}\right),$$

and since $t = \imath$ is a simple pole,

$$J(x) = 2\pi\imath \times \lim_{t\to\imath} \frac{e^{\imath xt}}{1+t^2}(t-\imath) = \frac{2\pi\imath e^{-x}}{2\imath} = \pi e^{-x}.$$

(ii) Evaluate $J(x)$ in sections

$$J(x) = \int_{-R}^{R} \frac{e^{\imath xt}}{1+t^2}\, dt \; + \; \int_{\Gamma_R} \frac{e^{\imath xt}}{1+t^2}\, dt.$$

Taking the limit $R \to \infty$,

$$\begin{aligned}
\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{\imath xt}}{1+t^2}\, dt &= \int_{-\infty}^{\infty} \frac{e^{\imath xt}}{1+t^2}\, dt \\
&= \int_{-\infty}^{0} \frac{e^{\imath xt}}{1+t^2}\, dt + \int_{0}^{\infty} \frac{e^{\imath xt}}{1+t^2}\, dt \\
\text{put } \; t = -s, \qquad &= \int_{0}^{\infty} \frac{e^{-\imath xs}}{1+s^2}\, ds + \int_{0}^{\infty} \frac{e^{\imath xt}}{1+t^2}\, dt \\
&= \int_{0}^{\infty} \frac{e^{-\imath xt} + e^{\imath xt}}{1+t^2}\, dt \; = \; 2\int_{0}^{\infty} \frac{\cos xt}{1+t^2}\, dt.
\end{aligned}$$

On Γ_R, $t = Re^{\imath\theta}$ with $0 < \theta < \pi$. Thus $e^{\imath xt} = e^{\imath xRe^{\imath\theta}} = e^{\imath xR(\cos\theta + \imath\sin\theta)}$,

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{e^{\imath xt}}{1+t^2}\, dt = \lim_{R\to\infty} \int_{0}^{\pi} \frac{e^{\imath xR(\cos\theta+\imath\sin\theta)}}{1+R^2 e^{2\imath\theta}}\, R\imath e^{\imath\theta} d\theta,$$

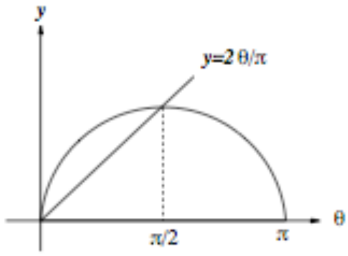
and

$$\left|\int_{\Gamma_R}\right| \le \int_0^{\pi} \left|\frac{e^{-xR\sin\theta}}{1+R^2e^{2\iota\theta}}\right| R\, d\theta.$$

Since R is sufficiently large, we have

$$\left|1+R^2e^{2\iota\theta}\right| \ge \left||1|-|R^2e^{2\iota\theta}|\right| = R^2-1.$$

Thus,



$$\begin{aligned}\left|\int_{\Gamma_R}\right| &\le \frac{2R}{R^2-1}\int_0^{\frac{\pi}{2}} e^{-xR\sin\theta}\, d\theta, \\ &\le \frac{2R}{R^2-1}\int_0^{\frac{\pi}{2}} e^{-xR\frac{2\theta}{\pi}}\, d\theta, \quad \sin\theta \ge \frac{2\theta}{\pi},\ 0\le\theta\le\frac{\pi}{2}, \\ &= \frac{2R}{R^2-1}\left[\frac{e^{-xR\frac{2\theta}{\pi}}}{-\frac{2}{\pi}xR}\right]_0^{\frac{\pi}{2}} \\ &= \frac{2R}{R^2-1}\cdot\frac{\pi}{2xR}\left(1-e^{-Rx}\right).\end{aligned}$$

$$\left|\int_{\Gamma_R}\right| \le \frac{\pi}{R^2-1}\cdot\frac{1-e^{-Rx}}{x} \to 0 \quad \text{as } R\to\infty.$$

Hence takeing limit as $R\to\infty$,

$$J(x) = 2\int_0^{\infty}\frac{\cos xt}{1+t^2}\, dt.$$

Combining results

$$2\int_0^{\infty}\frac{\cos xt}{1+t^2}\, dt = \pi e^{-x} \quad \Rightarrow \quad \int_0^{\infty}\frac{\cos xt}{1+t^2}\, dt = \frac{\pi}{2}e^{-x}.$$