國立高雄大學應用數學系

每月挑戰 020 解答

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1 a. $A \in M_{m\times n}(\mathbb{R})$. Show that

$$\lambda_{\max}^{\frac{1}{2}}(A^TA) = \max_{\|x\|_2\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2=1} \|Ax\|_2,$$

where $\lambda_{\max}(A^TA)$ is the maximal eigenvalue of A^TA.

Proof.

For any vector $x \in \mathbb{R}^n$ with $x \neq 0$, we have

$$\frac{\|Ax\|_2}{\|x\|_2} = \left\| A\frac{x}{\|x\|_2} \right\|_2.$$

Let $y = \frac{x}{\|x\|_2}$. Then $\|y\|_2 = 1$. Hence, we have

$$\max_{\|x\|_2\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2=1} \|Ax\|_2.$$

Now we show that $\lambda_{\max}^{\frac{1}{2}}(A^TA) = \max_{\|x\|_2=1} \|Ax\|_2$. Since A^TA is symmetric, there exists an orthogonal matrix $P \in M_{n\times n}(\mathbb{R})$ such that

$$A^TA = P^T \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}P, \tag{1}$$

where $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ are eigenvalues of A^TA. Since P is orthogonal, we have $\|Px\|_2 = \|x\|_2$ for any $x \in \mathbb{R}^n$. From (1), we know that for any unit vector $x \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n)^T = Px$, we have

$$\begin{aligned}\|Ax\|_2^2 &= x^T(A^TA)x = x^TP^T\mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}Px \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \leqslant \lambda_n.\end{aligned}$$

Hence, we obtain

$$\max_{\|x\|_2=1} \|Ax\|_2 \leqslant \lambda_n^{\frac{1}{2}} = \lambda_{\max}^{\frac{1}{2}}(A^TA). \tag{2}$$

Let u be a unit eigenvector of A^TA corresponding to eigenvalue λ_n. Then we have $\|Au\|_2 = \lambda_n^{\frac{1}{2}}$. From (2), we obtain

$$\max_{\|x\|_2=1} \|Ax\|_2 = \lambda_{\max}^{\frac{1}{2}}(A^TA).$$

答題優良名單：103 級翁嘉駿、博士班連威翔